An *n*th-order Darboux transformation for the one-dimensional time-dependent Schrödinger equation

2003 J. Phys. A: Math. Gen. 36 1615

(http://iopscience.iop.org/0305-4470/36/6/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 1615–1621 PII: S0305-4470(03)52517-3

An nth-order Darboux transformation for the one-dimensional time-dependent Schrödinger equation

Daniel J Arrigo and Fred Hickling

Department of Mathematics, University of Central Arkansas, Conway, AR 72035, USA

Received 14 August 2002, in final form 2 January 2003
Published 29 January 2003
Online at stacks.iop.org/JPhysA/36/1615

Abstract

Solutions for two one-dimensional Schrödinger equations are linked together using an nth-order Darboux transformation. The resulting coefficients for these Darboux transformations give rise to a system of n nonlinear partial differential equations. This system is shown to be equivalent to a matrix Burgers equation which is linearized using a generalized Hopf–Cole transformation. Solving this linear system provides the link between solutions of the two Schrödinger equations and shows how to factor an nth-order Darboux transformation into n first-order Darboux transformations.

PACS numbers: 02.30.Jr, 03.65.Ge

1. Introduction

One important tool for constructing exact solutions to a differential equation is to link solutions to the known solutions of another differential equation. Probably the simplest attempt using a differentiable transform was pioneered by Darboux [6] and is now known as a Darboux transformation. Using a linear differential operator, he constructed solutions of one ordinary differential equation in terms of another ordinary differential equation. Crum [5] further developed this idea by considering iterative applications of first-order Darboux transformations. He obtained convenient formulae for the Darboux transformations and their associated solutions. Extensions of Darboux transformation to linear partial differential equations and linear difference equations emerged in the work of Matveev [11, 12]. The impact of these transformations is illustrated in Matveev and Salle [10], where a wide variety of applications can be found. An excellent survey article of the crucial developments of Darboux transformations can be found in Rosu [13].

Darboux transformations have been applied to the one-dimensional Schrödinger equation. In a two-part paper, Bagrov and Samsonov [2] and Samsonov and Ovcharov [15] consider the time-independent Schrödinger equation and using a composition of n first-order Darboux transformations, determine new classes of solvable Schrödinger equations. These new classes of solvable equations have potentials which are related to the harmonic oscillator, Morse and effective Coulomb potentials. An extension to the time-dependent Schrödinger equation has also been given by Bagrov *et al* [3] but requires an additional assumption to determine the second-order Darboux transformation itself. In the case of linking solutions with the free particle equation, this assumption was lifted by Arrigo and Hickling [1] in their construction of the *n*th-order Darboux transformation. The work of Bagrov *et al* [3] suggests that an *n*th-order Darboux transformation is equivalent to iterating *n* first-order Darboux transformations. In this paper we address this question. Here, an *n*th-order Darboux transformation linking solutions of two different one-dimensional time-dependent Schrödinger equations is considered. This results in a system of *n* nonlinear partial differential equations for the coefficients of the transformation itself. Rewriting this system of equations in matrix form, and using a matrix Hopf–Cole transformation, linearizes the system, thus allowing for their solution. This process also shows how to factor (nonuniquely) an *n*th-order Darboux transformation into a composition of *n*th first-order transformations.

2. Darboux transformations

Consider the one-dimensional Schrödinger equation with a nonzero potential ($\hbar = 2m = 1$):

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + [f(t,x) + g(t,x)]\psi \tag{1}$$

where $f(t,x)$ and $g(t,x)$ are differentiable functions of their arguments. Introduce the Darboux transformation:

$$\psi = \frac{\partial^n\varphi}{\partial x^n} + \sum_{k=0}^{n-1} A_k(t,x)\frac{\partial^k\varphi}{\partial x^k} \tag{2}$$

where the $A_k(t,x)$ are differentiable functions of their arguments. Substituting (2) into (1) leads to

$$\begin{aligned}
&\mathrm{i}\frac{\partial^{n+1}\varphi}{\partial x^n\partial t} + \mathrm{i}\sum_{k=0}^{n-1}\left(A_k\frac{\partial^{k+1}\varphi}{\partial x^k\partial t} + \frac{\partial A_k}{\partial t}\frac{\partial^k\varphi}{\partial x^k}\right)\\
&\quad = -\frac{\partial^{n+2}\varphi}{\partial x^{n+2}} - \sum_{k=0}^{n-1}\left(A_k\frac{\partial^{k+2}\varphi}{\partial x^{k+2}} + 2\frac{\partial A_k}{\partial x}\frac{\partial^{k+1}\varphi}{\partial x^{k+1}} + \frac{\partial^2 A_k}{\partial x^2}\frac{\partial^k\varphi}{\partial x^k}\right)\\
&\qquad + (f+g)\left(\frac{\partial^n\varphi}{\partial x^n} + \sum_{k=0}^{n-1}A_k\frac{\partial^k\varphi}{\partial x^k}\right).
\end{aligned} \tag{3}$$

By requiring that φ satisfies the Schrödinger equation

$$\mathrm{i}\frac{\partial\varphi}{\partial t} = -\frac{\partial^2\varphi}{\partial x^2} + f(t,x)\varphi \tag{4}$$

and upon rearranging terms in (3), gives

$$\begin{aligned}
&-\left(g - 2\frac{\partial A_{n-1}}{\partial x}\right)\frac{\partial^n\varphi}{\partial x^n} + \sum_{k=1}^{n-1}\left(\mathrm{i}\frac{\partial A_k}{\partial t} + \frac{\partial^2 A_k}{\partial x^2} + 2\frac{\partial A_{k-1}}{\partial x} - gA_k\right)\frac{\partial^k\varphi}{\partial x^k}\\
&\quad + \sum_{k=1}^{n-1}\left(\binom{n}{k}\frac{\partial^{n-k}f}{\partial x^{n-k}} + \sum_{\ell=1}^{n-k-1}A_{k+\ell}\binom{\ell+k}{k}\frac{\partial^\ell f}{\partial x^\ell}\right)\frac{\partial^k\varphi}{\partial x^k}\\
&\quad \times\left(\mathrm{i}\frac{\partial A_0}{\partial t} + \frac{\partial^2 A_0}{\partial x^2} - gA_0 + \frac{\partial^n f}{\partial x^n} + \sum_{l=1}^{n-1}A_\ell\frac{\partial^\ell f}{\partial x^\ell}\right)\varphi = 0.
\end{aligned} \tag{5}$$

Since the solutions of the Schrödinger equation that appear in equation (5) can be varied arbitrarily, the coefficients involving φ and its higher order derivatives must vanish in order for (5) to be satisfied. This leads to the following system of partial differential equations for A_k and g:

$$\mathrm{i}\frac{\partial A_0}{\partial t} + \frac{\partial^2 A_0}{\partial x^2} - gA_0 + \frac{\partial^n f}{\partial x^n} + \sum_{l=1}^{n-1} A_k \frac{\partial^l f}{\partial x^l} = 0 \tag{6a}$$

$$\mathrm{i}\frac{\partial A_k}{\partial t} + \frac{\partial^2 A_k}{\partial x^2} + 2\frac{\partial A_{k-1}}{\partial x} - gA_k + \binom{n}{k}\frac{\partial^{n-k} f}{\partial x^{n-k}} + \sum_{\ell=1}^{n-k-1} A_{k+\ell}\binom{\ell+k}{k}\frac{\partial^\ell f}{\partial x^\ell} = 0$$

$$k = 1, 2, \ldots, n-1 \tag{6b}$$

$$g - 2\frac{\partial A_{n-1}}{\partial x} = 0. \tag{6c}$$

Further, eliminating g from (6a) and (6b) using (6c) yields

$$\mathrm{i}\frac{\partial A_0}{\partial t} + \frac{\partial^2 A_0}{\partial x^2} - 2A_0\frac{\partial A_{n-1}}{\partial x} + \frac{\partial^n f}{\partial x^n} + \sum_{l=1}^{n-1} A_l \frac{\partial^l f}{\partial x^l} = 0 \tag{7a}$$

$$\mathrm{i}\frac{\partial A_k}{\partial t} + \frac{\partial^2 A_k}{\partial x^2} + 2\frac{\partial A_{k-1}}{\partial x} - 2A_k\frac{\partial A_{n-1}}{\partial x} + \binom{n}{k}\frac{\partial^{n-k} f}{\partial x^{n-k}} + \sum_{l=1}^{n-k-1} A_{k+\ell}\binom{\ell+k}{k}\frac{\partial^\ell f}{\partial x^\ell} = 0$$

$$k = 1, 2, \ldots, n-1. \tag{7b}$$

This system of PDEs can conveniently be written in matrix Burgers form

$$\mathrm{i}\Omega_t - 2\Omega_x\Omega + \Omega_{xx} + \Omega F - F\Omega + F_x = 0 \tag{8}$$

where Ω is the $n \times n$ matrix given by

$$\Omega = \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1 \\ A_0 & A_1 & \cdots & A_{n-2} & A_{n-1} \end{bmatrix} \tag{9}$$

and F is the $n \times n$ matrix whose entries are given by

$$F_{i,j} = \begin{cases} f & i = j = 1 \\ \binom{i-1}{j-1}\frac{\partial^{i-j} f}{\partial^{i-j} x} & i, j = 2, 3, \ldots, n \quad j \leqslant i \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Following Levi *et al* [9], introduce the matrix Hopf–Cole transformation:

$$\Omega = -\frac{\partial \Phi}{\partial x}\Phi^{-1}. \tag{11}$$

Substituting (11) into equation (8) leads to the linear matrix Schrödinger equation

$$\mathrm{i}\frac{\partial \Phi}{\partial t} = -\frac{\partial^2 \Phi}{\partial x^2} + F\Phi \tag{12}$$

from which we deduce that the entries ϕ_{ij} of the matrix Φ satisfy the following:

$$\mathrm{i}\frac{\partial\phi_{1,\ell}}{\partial t}+\frac{\partial^2\phi_{1,\ell}}{\partial x^2}=f\phi_{1,\ell} \qquad \ell=1,2,\ldots,n \tag{13a}$$

$$\mathrm{i}\frac{\partial\phi_{k,\ell}}{\partial t}+\frac{\partial^2\phi_{k,\ell}}{\partial x^2}=\sum_{m=1}^{k}\binom{k-1}{m-1}\frac{\partial^{k-m}f}{\partial x^{k-m}}\phi_{m,\ell} \qquad k,\ell=2,3,\ldots,n. \tag{13b}$$

Using (11) or more specifically

$$\Omega\Phi=-\frac{\partial\Phi}{\partial x} \tag{11$'$}$$

with Ω as given in (9), leads to, on a component by component comparison of the entries of the matrices in (11′),

$$\phi_{k+1,\ell}=\frac{\partial\phi_{k,\ell}}{\partial x} \tag{14a}$$

$$\sum_{k=1}^{n-1}A_{k-1}\phi_{k,\ell}=-\frac{\partial\phi_{n+1,\ell}}{\partial x}. \tag{14b}$$

If we denote $\phi_{1\ell}=\omega_\ell$, $l=1,2,\ldots,n$, then from (13a) we deduce that ω_ℓ are solutions of the Schrödinger equation (4). From (14a) we obtain

$$\phi_{k,\ell}=\frac{\partial^{k-1}\omega_\ell}{\partial x^{k-1}} \tag{15}$$

which in turn gives (13b) as

$$\mathrm{i}\frac{\partial^{k-1}}{\partial x^{k-1}}\left(\frac{\partial\omega_\ell}{\partial t}\right)+\frac{\partial^{k-1}}{\partial x^{k-1}}\left(\frac{\partial^2\omega_\ell}{\partial x^2}\right)=\sum_{m=1}^{k}\binom{k-1}{m-1}\frac{\partial^{k-m}f}{\partial x^{k-m}}\frac{\partial^{k-1}\omega_\ell}{\partial x^{k-1}}. \tag{16}$$

This upon rearrangement leads to

$$\mathrm{i}\frac{\partial^{k-1}}{\partial x^{k-1}}\left(\frac{\partial\omega_\ell}{\partial t}\right)+\frac{\partial^{k-1}}{\partial x^{k-1}}\left(\frac{\partial^2\omega_\ell}{\partial x^2}\right)=\frac{\partial^{k-1}(f\omega_\ell)}{\partial x^{k-1}} \tag{17}$$

which is satisfied because ω_ℓ satisfies (4). Finally, from (14b), we obtain

$$\frac{\partial^n\omega_\ell}{\partial x^n}+\sum_{m=0}^{n-1}A_m\frac{\partial^m\omega_\ell}{\partial x^m}=0. \tag{18}$$

The system of equations (18) can be solved for the A_m giving

$$A_m=(-1)^{n-m}\frac{W_{\widehat{m}}}{W} \tag{19}$$

where

$$W_{\widehat{m}}=\begin{vmatrix}\omega_1 & \omega_2 & \cdots & \omega_{n-1} & \omega_n\\ \partial_x\omega_1 & \partial_x\omega_2 & \cdots & \partial_x\omega_{n-1} & \partial_x\omega_n\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \partial_x^{m-1}\omega_1 & \partial_x^{m-1}\omega_2 & \cdots & \partial_x^{m-1}\omega_{n-1} & \partial_x^{m-1}\omega_n\\ \partial_x^{m+1}\omega_1 & \partial_x^{m+1}\omega_2 & \cdots & \partial_x^{m+1}\omega_{n-1} & \partial_x^{m+1}\omega_n\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \partial_x^{n}\omega_1 & \partial_x^{n}\omega_2 & \cdots & \partial_x^{n}\omega_{n-1} & \partial_x^{n}\omega_n\end{vmatrix} \tag{20}$$

and

$$W = \begin{vmatrix} \omega_1 & \cdots & \omega_n \\ \vdots & \ddots & \vdots \\ \partial_x^{n-1}\omega_1 & \cdots & \partial_x^{n-1}\omega_n \end{vmatrix} \tag{21}$$

where, as usual, $|\ |$ represents the determinant and for convenience, the notation $\partial_x^k \equiv \frac{\partial^k}{\partial x^k}$ has been used. Thus, it follows that if φ is a solution of Schrödinger's equation (4) with potential $f(t, x)$, then

$$\psi = \frac{\partial^n \varphi}{\partial x^n} + \sum_{k=0}^{n-1} (-1)^{n-k} \frac{W_{\widehat{k}}}{W} \frac{\partial^k \varphi}{\partial x^k} \tag{22}$$

is a solution to Schrödinger's equation (1) with potential $f(t, x) + g(t, x)$, when $g(t, x)$ is of the form

$$g(t, x) = -2\frac{W_{\widehat{n-1}}}{W} = -2\frac{\partial^2}{\partial x^2}(\ln W). \tag{23}$$

A more compact form of the solution given in (22) is provided by

$$\psi = \frac{W_\varphi}{W} \tag{24}$$

where $W_\varphi = \det(\text{wronskian}(\omega_1, \omega_2, \ldots, \omega_n, \varphi))$ and where each of the ω_ℓ and φ are solutions of (4). It should be noted that if φ is a linear combination of the ω_i then the solution ψ is trivial.

3. Iterated and nth-order equivalence

The previous section saw the construction of the solution

$$\psi = \frac{\partial^n \varphi}{\partial x^n} + \sum_{k=0}^{n-1} (-1)^{n-k} \frac{W_{\widehat{k}}}{W} \frac{\partial^k \varphi}{\partial x^k} \tag{22}$$

for Schrödinger's equation with potential $f(t, x) + g(t, x)$ where $W_{\widehat{k}}$, W and $g(t, x)$ were given in (20), (21) and (23) respectively, provided φ and ω_ℓ appearing in W and $W_{\widehat{k}}$ all satisfy Schrödinger's equation with potential $f(t, x)$.

We now focus our attention on the operator

$$\frac{\partial^n}{\partial x^n} + \sum_{k=0}^{n-1} (-1)^{n-k} \frac{W_{\widehat{k}}}{W} \frac{\partial^k}{\partial x^k}. \tag{25}$$

The literature usually sees an nth-order Darboux transformation given as a composition of n first-order Darboux transformations (see, e.g., [14]). In fact, the operator given in (25) is factorable and can be given as

$$\frac{\partial^n}{\partial x^n} + \sum_{k=0}^{n-1} (-1)^{n-k} \frac{W_{\widehat{k}}}{W} \frac{\partial^k}{\partial x^k} = \prod_{\ell=1}^{n} \left(\frac{\partial}{\partial x} - \frac{1}{\Phi_\ell} \frac{\partial(\Phi_\ell)}{\partial x} \right) \tag{26}$$

where $W_{\widehat{k}}$ is given in (20),

$$\Phi_\ell = \frac{W^\ell}{W^{\ell-1}} \tag{27}$$

and W^ℓ is given by

$$W^\ell = \det \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_\ell \\ \partial_x\omega_1 & \partial_x\omega_2 & \cdots & \partial_x\omega_\ell \\ \vdots & \vdots & \ddots & \vdots \\ \partial_x^{\ell-1}\omega_1 & \partial_x^{\ell-1}\omega_2 & \cdots & \partial_x^{\ell-1}\omega_\ell \end{bmatrix} \tag{28}$$

with $W^0 = 1$. By (24), $\Phi_\ell = \frac{W^\ell}{W^{\ell-1}}$ is a solution to the equation

$$\mathrm{i}\frac{\partial\varphi}{\partial t} = -\frac{\partial^2\varphi}{\partial x^2} + (f(t,x) + g_{\ell-1}(t,x))\varphi \tag{29}$$

where $g_{\ell-1} = \frac{\partial^2 \ln(W^{\ell-1})}{\partial^2 x}$, and the operator

$$\frac{\partial}{\partial x} - \frac{1}{\Phi_\ell}\frac{\partial(\Phi_\ell)}{\partial x} \tag{30}$$

transforms solutions of (29) into solutions of

$$\mathrm{i}\frac{\partial\varphi}{\partial t} = -\frac{\partial^2\varphi}{\partial x^2} + (f(t,x) + g_\ell(t,x))\varphi. \tag{31}$$

Thus the right-hand side of (26) is the composition of transformations which take solutions of $\mathrm{i}\frac{\partial\varphi}{\partial t} = -\frac{\partial^2\varphi}{\partial x^2} + f(t,x)\varphi$ to solutions of $\mathrm{i}\frac{\partial\varphi}{\partial t} = -\frac{\partial^2\varphi}{\partial x^2} + (f(t,x) + g_n(t,x))\varphi$ by going through solutions to the intermediate equations (29) and (31) for $1 \leqslant \ell \leqslant n$. Factorizations of ordinary differential operators with coefficients of one variable and a similar structure to (25) have appeared in [4]; however, the coefficeints of the operator (25) are of two variables.

It is interesting to note that the factorization of the nth-order Darboux operator (25) given in (26) is not unique. Simply reordering the solutions $\{\omega_1, \ldots, \omega_n\}$ and then applying the same factorization technique results in a different factorization. In fact, for a generic nth-order Darboux operator there is an $(n-1)!$-dimensional set of distinct factorization. This follows by induction on the order of the operator, that linear combinations of a solution is again a solution, but that rescalling a solution ω_k or a linear combination of solutions does not change W^ℓ.

4. Conclusion

This paper constructs a link between Schrödinger equations with different nonzero potentials using an nth-order Darboux transformation. A system of n nonlinear partial differential equations is constructed for the coefficients of the Darboux transformation. Using a matrix Hopf–Cole transformation, this system is solved. This improves and generalizes the earlier work of Samsonov and Ovcharov [15] who required an assumption to show that the composition of two first-order Darboux transformations for the time-dependent Schrödinger equation was equivalent to a second-order Darboux transformation. They further conjectured the equivalence of an nth-order Darboux transformation and a composition of n first-order Darboux transformations. In this paper we have shown this conjecture to be true and give an explicit (though nonunique) factorization.

Acknowledgments

The authors gratefully acknowledge the financial support of the University Research Council at the University of Central Arkansas. We would also like to thank the reviewers for their helpful suggestions.

References

[1] Arrigo D J and Hickling F 2002 *J. Phys. A: Math. Gen.* **V35** 389
[2] Bargrov V G and Samsonov B F 1995 *Theor. Math. Phys.* **104** 1051
Bargrov V G and Samsonov B F 1997 *Pramana* **49** 563
[3] Bagrov V G, Samsonov B F and Shekoyan L A 1995 *Russ. Phys. J.* **38** 706
[4] Berkovitch L V 2001 The integration of ordinary differential equations: factorizations and transformations *Pure Math.*/0107037 Samara State University
[5] Crum M 1955 *Q. J. Math.* **6** 121–8
[6] Darboux G 1882 *C. R. Acad. Sci.* **94** 1456
[7] Englefield M J 1987 *J. Math. Phys. A: Math. Gen.* **20** 593
[8] Flügge S 1994 *Practical Quantum Mechanics* 2nd edn (Berlin: Springer) p 89, 94
[9] Levi D, Ragnisco O and Bruschi M 1983 *Nuovo Cimento* **74** 33
[10] Matveev V and Salle M A 1991 *Darboux Transformations and Solitons (Springer Series in Nonlinear Dynamics)* (Berlin: Springer)
[11] Matveev V 1979 *Lett. Math. Phys.* **3** 213–6
[12] Matveev V 1979 *Lett. Math. Phys.* **3** 217–22
[13] Rosu H C 1998 *Preprint* quant-ph/9809056
[14] Samonsov B F 1999 *Phys. Lett.* A **263** 274
[15] Samonsov B F and Ovcharov I N 1995 *Russ. Phys. J.* **38** 765